we obtain that the condition for reversal of waves being propagated to the right $\partial C / \partial t > 0$ is satisfied for $\pi / 2 < \theta < \theta^2$ $(2\pi - \theta_2 < \theta < 3\pi / 2)$ for fast plastic waves. Therefore, taking account of thermal effects results in the need to consider jumps in the plastic domain in contrast to the uncoupled model [6]. In those cases when the jump is of sufficiently small intensity, for example, if it originates because of reversal of the simple wave and $\theta_2 - \pi / 2$ is small (this quantity is on the order of 10^{-3} for steel), the relationships between the quantities in the appropriate simple wave can be used as approximate conditions on the jump, and the rate of propagation of the discontinuity can approximately be considered equal to the average of the values of the appropriate characteristic velocities ahead of and behind the jump.

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ON THE HAMILTON-JACOBI METHOD FOR NONHOLONOMIC SYSTEMS

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The progress made in the theory of integration of equations of motion of holonomic systems naturally leads to attempts to extend the basic assumptions of this theory to nonholonomic systems, or at least to establish the conditions for their applicability to nonholonomic systems. Problems of this type were the subject of many papers by various authors. In particular, numerous attempts were made to extend the Hamilton-Jacobi method of integration to the nonholonomic systems (see [1]). Below we discuss the problems relevant to the latter problem.

1. The first attempt of generalizing the Hamilton-Jacobi method to cover the nonholonomic systems was made in [1]. Assuming that the differential constraints imposed on the system are linear, the author obtains the equations of motion in the form (*)

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} + R_i \tag{1.1}$$

where R_i are either zero or represent quadratic functions of the generalized impulses, and formulates the following theorem: the Jacobi theorem can be used for a nonholonomic system if and only if the normal Jacobi equation can be supplemented with a function φ such, that R_i are its partial derivatives with respect to the coordinate x^i. The author also assumes that φ is a function of the coordinates and impulses of the system. No examples are given to illustrate the theorem. Paper [3] gives a similar theorem for nonholonomic systems with nonlinear constraints $f_\sigma(t, x^i, \dot{x}^i) = 0$. The equations of motion of these systems are written with the help of undetermined multipliers in the form (1.1) where

$$R_i = \lambda_\sigma \frac{\partial f_\sigma}{\partial \dot{x}^i}$$

under the assumption that the constraints are of the Appel-Chetaev type. All this having been said, it can easily be shown that the theorem quoted above holds in neither of these two cases.

Indeed, suppose a function φ exists such that

$$R_i = -\frac{\partial \varphi}{\partial x^i} \tag{1.2}$$

and

$$\frac{\partial V}{\partial a^i} = b_i = \text{const}\ (a^i = \text{const}), \qquad \frac{\partial V}{\partial x^i} = p_i \tag{1.3}$$

yield a complete system of solutions of (1.1) provided that $V(t, x^i, a^i)$ is a general integral of the equation

$$\frac{\partial V}{\partial t} + H + \varphi = 0 \tag{1.4}$$

We shall show that these assumptions lead to a contradiction. Differentiating both parts of the first group of equations (1.3) with respect to time and taking into account the first group of equations in (1.1), we obtain

$$\frac{\partial^2 V}{\partial a^i \partial t} + \frac{\partial^2 V}{\partial a^i \partial x^j}\frac{\partial H}{\partial p_j} \equiv 0 \tag{1.5}$$

Differentiating now the left hand side of (1.4) with respect to a^i under the assumption that the general integral V has been inserted into that expression and taking into account the second group of equations (1.3), we obtain

$$\frac{\partial^2 V}{\partial t \partial a^i} + \frac{\partial (H + \varphi)}{\partial p_j}\frac{\partial^2 V}{\partial x^j \partial a^i} \equiv 0 \tag{1.6}$$

Subtracting (1.6) from (1.5) yields the following system of n linear homogeneous equations

$$\frac{\partial \varphi}{\partial p_j}\frac{\partial^2 V}{\partial x^j \partial a^i} \equiv 0 \tag{1.7}$$

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*) The indices used assume the following values: $i, j = 1, \ldots, n$; $k, s = 0, 1, \ldots, n$; $\alpha, \beta = 1, \ldots, m$; $\nu, \gamma = 0, 1, \ldots, m$; $\sigma = m + 1, \ldots, n$. Repeated indices denote summation.

in n functions $d\varphi/dp_j$. But V is a general integral of (1.4), hence

$$\left|\frac{\partial^2 V}{\partial x^j \partial a^i}\right| \neq 0$$

and from (1.7) we obtain

$$\frac{\partial \varphi}{\partial p_j} \equiv 0 \tag{1.8}$$

i.e. the function φ contains no impulses p_j. Differentiating (1.2) and (1.8) with respect to p_j and x^i respectively, we find that

$$\frac{\partial R_i}{\partial p_j} \equiv 0$$

i.e. R_i also contain no impulses p_j.

But the quantities R_i depend essentially on the impulses for both the linear and the nonlinear constraints. The contradiction obtained proves the statement given above.

2. In [4] we find a perfectly correct and fully substantiated assertion that the Hamilton-Jacobi method cannot, in general, be applied to nonholonomic systems, although some nonholonomic systems exist, the equations of motion of which can either be directly integrated using the Hamilton-Jacobi method, or the method can be applied after the equations of motion have been somewhat transformed. The paper gives also the necessary and sufficient conditions for the partial applicability (the term is explained at the end of Sect. 2) of the method to nonholonomic systems with linear differential constraints, the equations of motion of which are written in the form of the Ferrers equations with constraint multipliers. We shall now derive the analogous conditions for a more general case of the equations of motion of nonholonomic systems, written in the Poincaré-Chetaev variables.

Let the position of the mechanical system be determined by the variables $x^1, \ldots, x^n$. We define the infinitesimal displacements of the system by means of a set of independent operators

$$X_\nu = \xi_\nu^{\ s}(x^k)\frac{\partial}{\partial x^s}, \qquad x^\circ = t, \qquad \xi_0^{\ \circ} = 1, \qquad \xi_\alpha^{\ \circ} = 0 \tag{2.1}$$

in such a manner that the variation of an arbitrary function $F(x^k)$ over a real (possible) displacement of the system is determined by the equation

$$dF = \eta^\nu X_\nu F dx^\circ \quad (\delta F = \omega^\alpha X_\alpha F), \quad \eta^\circ = x^{\cdot\circ} = 1 \tag{2.2}$$

The independent parameters η^α (ω^α) characterize the real (possible) displacements of the system and their number is equal to the number of degrees of freedom of the system. We assume that the operator set (2.1) is not closed. This means that the corresponding system of Pfaffian forms is not integrable. Since these forms are equated to zero, they represent the differential nonintegrable constraints of the system, and under the assumption made the system considered is therefore nonholonomic. Applying the formulas (2.1) and (2.2) for the functions $F^k = x^k$ we obtain

$$x^{\cdot k} = \xi_\nu^{\ k}\eta^\nu \tag{2.3}$$

Since the operators (2.1) are independent, it is possible to cut out from the matrix $\|\xi_\nu^{\ k}\|$, a quadratic $(m+1)\times(m+1)$ matrix whose determinant is not zero. Let $\|\xi_\nu^{\ \gamma}\|$ be this matrix. Then the first $m+1$ equations of (2.3) yield

$$\eta^\nu = b_\gamma^{\ \nu} x^{\cdot\gamma} \tag{2.4}$$

Inserting these expressions into the remaining $n-m$ equations of (2.3), we obtain the following equations for the differential nonintegrable constraints of the system

$$x^{\cdot\sigma} = B_\gamma{}^\sigma x^{\cdot\gamma} \tag{2.5}$$

Let $L(x^k, x^{\cdot i})$ be the Lagrangian function for a constraint-free system and $L'(x^k, \eta^\alpha)$ a Lagrangian function expressed with the help of (2.3) in terms of the independent parameters. Introducing the Hamiltonian function

$$H' = y_\alpha \eta^\alpha - L', \qquad y_\alpha = \frac{\partial L'}{\partial \eta^\alpha}$$

we can write the equations of motion of the system in the form (*)

$$y_\alpha^{\cdot} = -X_\alpha H' + \eta^\nu \left[\Omega^\sigma_{\nu\alpha}\left(\frac{\partial L}{\partial x^{\cdot\sigma}}\right)^* - \Gamma^\beta_{\nu\alpha} y_\beta\right], \qquad \eta^\alpha = \frac{\partial H'}{\partial y_\alpha} \tag{2.6}$$

$$\Omega^\sigma_{\nu\alpha} = \xi_\alpha{}^\gamma X_\nu B_\gamma{}^\sigma - \xi_\nu{}^\gamma X_\alpha B_\gamma{}^\sigma \tag{2.7}$$

$$\Gamma^\beta_{\nu\alpha} = b_\gamma{}^\beta (X_\alpha \xi_\nu{}^\gamma - X_\nu \xi_\alpha{}^\gamma)$$

which can easily be transformed into

$$y_\alpha^{\cdot} = -X_\alpha H' + \eta^\nu (X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i)\left(\frac{\partial L}{\partial x^{\cdot i}}\right)^*, \qquad \eta^\alpha = \frac{\partial H'}{\partial y_\alpha} \tag{2.8}$$

The notation with an asterisk $(\ )^*$ indicates that the corresponding functions are written in the variables x^i and y_α. Equations (2.6) and (2.8) differ from the well known equations of motion in nonholonomic coordinates [1] only in the manner of notation. The method of nonholonomic coordinates and the method of the Poincaré-Chetaev variables are in fact identical, only the terminology is different. In this paper we prefer to use the Poincaré-Chetaev terminology, since their concepts have a definite mechanical meaning.

Let us write the equation

$$X_0 V + H'(x^k, X_\alpha V) = 0 \tag{2.9}$$

(which is due to Chetaev [5]) and prove the following theorem.

Theorem. If an integral $V(x^k, a^\lambda)$ of Eq. (2.9) containing a number of arbitrary nonadditive constants a^λ $(\lambda = 1, \ldots, l)$ and satisfying the conditions

$$(X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i)\left[\left(\frac{\partial L}{\partial x^{\cdot i}}\right)^* - \frac{\partial V}{\partial x^i}\right] = 0 \tag{2.10}$$

exists, then the equations

$$y_\alpha = X_\alpha V, \qquad \frac{\partial V}{\partial a^\lambda} = b_\lambda = \text{const} \tag{2.11}$$

represent the integrals of canonical equations of motion. The quantities y_α appearing in the functions $(\partial L/\partial x^{\cdot i})^*$ in (2.10), are supposed to be replaced by $X_\alpha V$ in accordance with the first group of equations (2.11).

Proof. Assume that the integral V of (2.9) satisfying the conditions of the theorem has been found. We shall show that the time derivatives of the first group of equations in (2.11) can be reduced to identities by virtue of (2.8) (or (2.6)), of (2.10) and of the equations themselves. Differentiating both parts of the equations indicated, with respect to time, and taking into account the first group of equations in (2.8) as well as the relation

$$(X_\nu, X_\alpha) V = (X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i) \frac{\partial V}{\partial x^i}$$

*) See the dissertation of E. Kh. Naziev, Some Problems of the Analytic Dynamics, Izd. MGU, 1969.

we obtain

$$X_\alpha H' + \eta^\nu \left\{ X_\alpha X_\nu V + (X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i) \left[\frac{\partial V}{\partial x^i} - \left(\frac{\partial L}{\partial \dot{x}^i} \right)^* \right] \right\} = 0$$

Since V is the integral of (2.9) satisfying the conditions of the theorem, the second terms within the brackets in each of the equations obtained vanish identically. Therefore, using the second group of equations (2.8) and the first group of equations (2.11), we obtain

$$X_\alpha X_0 V + X_\alpha H' + \frac{\partial H'}{\partial (X_\beta V)} X_\alpha X_\beta V = 0$$

But the latter are identities since they are obtained by applying the operator X_α to the left-hand part of (2.9) into which the known integral V has been inserted; QED.

We shall now show that, when the conditions of the theorem hold, the time derivatives of the left-hand parts of the second group of equations in (2.11) vanish identically by virtue of (2.8) and the first group of equations in (2.11). By (2.8) we have

$$\frac{d}{dx^0} \frac{\partial V}{\partial a^\lambda} = X_0 \frac{\partial V}{\partial a^\lambda} + \frac{\partial H'}{\partial y_\beta} X_\beta \frac{\partial V}{\partial a^\lambda}$$

Since V is the integral of (2.9) satisfying the conditions of the theorem, the first group of equations in (2.11) holds in accordance with the first part of the proof. Therefore, if we assume that the order in which the operations X_ν and $\partial/\partial a^\lambda$ are performed is immaterial, the previous equations can be rewritten in the form

$$\frac{d}{dx^0} \frac{\partial V}{\partial a^\lambda} = \frac{\partial}{\partial a^\lambda} X_0 V + \frac{\partial H'}{\partial (X_\beta V)} \frac{\partial}{\partial a^\lambda} X_\beta V$$

But the right-hand sides of the latter equations are identically zero since they represent the partial derivatives with respect to a^λ of the left-hand part of (2.9) in which V has been replaced by the known expression. Therefore

$$\frac{d}{dx^0} \frac{\partial V}{\partial a^\lambda} \equiv 0$$

QED.

It can be shown that Eqs. (2.10) are satisfied identically if the system is holonomic (the set of operators (2.1) is closed). Then the knowledge of the general integral of (2.9) makes it possible to obtain a general solution of the problem. If on the other hand the system is nonholonomic, then the problem reduces to investigating the consistency of the system (2.9) and (2.10) of first order partial differential equations. In many cases the system is not consistent. When the consistence occurs, the general solution of the problem cannot, generally speaking, be obtained. One can therefore speak only of partial applicability of the Hamilton-Jacobi method of integration to nonholonomic systems.

3. Nonholonomic systems whose equations of motion can be directly integrated using the Hamilton-Jacobi method, exist. These are the Chaplygin-type systems for which

$$\Omega^\sigma_{\nu\alpha} \left(\frac{\partial L}{\partial \dot{x}^\sigma} \right)^* \equiv 0$$

in the equations of motion (2.6). This may occur when $\Omega^\sigma_{\nu\alpha} \neq 0$ for a certain structure of the Lagrangian function L. We shall show, that in such cases the system (2.9), (2.10) is always consistent. Indeed, in such a case (2.10) can be reduced to the form

$$(\Gamma^\beta_{\nu\alpha} \xi_\beta{}^\sigma + X_\nu \xi_\alpha{}^\sigma - X_\alpha \xi_\nu{}^\sigma) \frac{\partial V}{\partial x^\sigma} = 0$$

The latter equations are satisfied by any function V which is independent of variables x^σ. Since these variables do not enter the coefficients of (2.9), we can neglect the terms in the operators (2.1) containing partial derivatives with respect to these variables and write (2.9) in the form

$$\xi_0{}^\nu \frac{\partial V}{\partial x^\nu} + H'\left(x^\gamma, \xi_\alpha{}^\nu \frac{\partial V}{\partial x^\nu}\right) = 0$$

After the general integral of this equation has been determined, solution of the problem can be reduced to algebraic operations and to determination of the functions $x^\sigma(t)$ from (2.5) by means of quadratures. Obviously the solution thus obtained is a general one.

Note. It should not be presumed that an analogous situation exists for the Chaplygin-type equations in the case when in Eqs.(2.8)

$$(X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i) \left(\frac{\partial L}{\partial x^{\cdot i}}\right)^* = 0$$

In this case Eqs.(2.10) are written in the form

$$(X_\nu \xi_\alpha{}^i - X_\alpha \xi_\nu{}^i) \frac{\partial V}{\partial x^i} = 0$$

and the compatibility of the latter with (2.9) is not implied anywhere.

4. Example. Let us consider a motion of a sharp-edged homogeneous disc of radius a on a horizontal plane. Let $O\xi\eta\zeta$ be the fixed coordinate system, the $O\xi$- and $O\eta$-axes lying in the plane and $O\zeta$-axis pointing vertically upwards. We adopt the O_1xyz system as the moving coordinate system, with its origin at the disc center. The O_1x-axis lies in the disc plane and is parallel to the line of intersection of this plane with the $O\xi\eta$ plane, the O_1y-axis is collinear with the diameter of the disc passing upwards through its point of contact with the plane, and the O_1z-axis is perpendicular to the disc plane. The following variables serve to determine the position of the disc: the ξ, η and ζ coordinates of the center of the disc in the $O\xi\eta\zeta$ system, the angle θ between the O_1z-axis and the vertical passing through the disc center, the angle ψ between the $O\xi$-axis and the line of intersection of the disc plane with the $O\xi\eta$ plane and the angle φ formed by some fixed radius of the disc and the O_1x-axis.

The Lagrangian function and the equations of differential constraints have the form

$$L_0 = {}^1\!/_2 m (\xi^{\cdot 2} + \eta^{\cdot 2} + \zeta^{\cdot 2}) + {}^1\!/_2 A (\theta^{\cdot 2} + \psi^{\cdot 2} \sin^2\theta) + {}^1\!/_2 C (\varphi^\cdot + \psi^\cdot \cos\theta)^2 - mga \sin\theta$$

$$\xi^\cdot = a(\theta^\cdot \sin\psi \sin\theta - \psi^\cdot \cos\psi \cos\theta - \varphi^\cdot \cos\psi)$$

$$\eta^\cdot = -a(\theta^\cdot \cos\psi \sin\theta + \psi^\cdot \sin\psi \cos\theta + \varphi^\cdot \sin\psi)$$

$$\zeta^\cdot = a\theta^\cdot \cos\theta$$

The last constraint equation can be integrated. Eliminating $\zeta^\cdot$ from L_0 we obtain

$$L = {}^1\!/_2 m (\xi^{\cdot 2} + \eta^{\cdot 2}) + {}^1\!/_2 (A + ma^2 \cos^2\theta)\theta^{\cdot 2} + {}^1\!/_2 A\psi^{\cdot 2}\sin^2\theta + {}^1\!/_2 C(\varphi^\cdot + \psi^\cdot \cos\theta)^2 - mga\sin\theta$$

Let us choose the following quantities as the parameters defining the real displacement of the disc

$$\eta^1 = \theta^\cdot, \quad \eta^2 = \psi^\cdot \sin\theta, \quad \eta^3 = \varphi^\cdot + \psi^\cdot \cos\theta$$

Then

$$x^{\cdot 1} = \theta^\cdot = \eta^1, \quad x^{\cdot 2} = \psi^\cdot = \frac{\eta^2}{\sin\theta}, \quad x^{\cdot 3} = \varphi^\cdot = \eta^3 - \eta^2 \operatorname{ctg}\theta$$

$$x^{\cdot 4} = \xi^\cdot = a(\eta^1 \sin\psi \sin\theta - \eta^3 \cos\psi)$$

$$x^{\cdot 5} = \eta^\cdot = -a(\eta^1 \cos\psi \sin\theta + \eta^3 \sin\psi)$$

The infinitesimal disc displacement operators have the form

$$X_1 = \frac{\partial}{\partial\theta} + a\sin\psi\sin\theta\frac{\partial}{\partial\xi} - \cos\psi\sin\theta\frac{\partial}{\partial\eta}$$
$$X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}$$
$$X_3 = \frac{\partial}{\partial\varphi} - a\cos\psi\frac{\partial}{\partial\xi} - a\sin\psi\frac{\partial}{\partial\eta}$$

In the independent variables the Lagrangian function L becomes

$$L' = \tfrac{1}{2}(A + ma^2)(\eta^1)^2 + \tfrac{1}{2}A(\eta^2)^2 + \tfrac{1}{2}(C + ma^2)(\eta^3)^2 - mga\sin\theta$$
$$y_1 = \frac{\partial L'}{\partial\eta^1} = (A + ma^2)\eta^1, \quad y_2 = \frac{\partial L'}{\partial\eta^2} = A\eta^2, \quad y_3 = \frac{\partial L'}{\partial\eta^3} = (C + ma^2)\eta^3$$

and the Hamiltonian function has the form

$$H' = \frac{1}{2}\left(\frac{y_1^2}{A + ma^2} + \frac{y_2^2}{A} + \frac{y_3^2}{C + ma^2}\right) + mga\sin\theta$$

Since the system under consideration is scleronomous, the function $V = -ht + W$ satisfies (2.9). The partial differential equation in W is written in the form

$$\frac{1}{A + ma^2}\left(\frac{\partial W}{\partial\theta} + a\sin\psi\sin\theta\frac{\partial W}{\partial\xi} - a\cos\psi\sin\theta\frac{\partial W}{\partial\eta}\right)^2 + \frac{1}{A}\left(\frac{1}{\sin\theta}\frac{\partial W}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial W}{\partial\varphi}\right)^2 +$$
$$\frac{1}{C + ma^2}\left(\frac{\partial W}{\partial\varphi} - a\cos\varphi\frac{\partial W}{\partial\xi} - a\sin\psi\frac{\partial W}{\partial\eta}\right)^2 + 2mga\sin\theta = 2h \qquad (4.1)$$

The system (2.10) consists of a single equation

$$\sin\psi\frac{\partial W}{\partial\xi} - \cos\psi\frac{\partial W}{\partial\eta} - \frac{ma\sin\theta}{A + ma^2\cos^2\theta}\frac{\partial W}{\partial\theta} = 0 \qquad (4.2)$$

The other two equations are identities and are, in general, inconsistent.

Let us consider a particular case when $\partial W/\partial\theta = 0$, corresponding to the motion in which $\theta^{\cdot} = 0, \theta = \lambda = \mathrm{const}$. Then the system (4.1), (4.2) is satisfied by the function W independent of ξ and η and the problem reduces to finding the general integral of the equation

$$\frac{1}{A}\left(\frac{1}{\sin\lambda}\frac{\partial W}{\partial\psi} - \operatorname{ctg}\lambda\frac{\partial W}{\partial\varphi}\right)^2 + \frac{1}{C + ma^2}\left(\frac{\partial W}{\partial\varphi}\right)^2 = 2(h - mga\sin\lambda)$$

This integral has the form $W = \alpha\psi + \beta\varphi$ where α and β are constants connected between themselves and related to the constant h by the equation

$$\frac{1}{A}\left(\frac{\alpha}{\sin\lambda} - \beta\operatorname{ctg}\lambda\right)^2 + \frac{\beta^2}{C + ma^2} = 2(h - mga\sin\lambda)$$

and

$$\psi + \frac{\partial\beta}{\partial\alpha}\varphi = \alpha_1, \qquad t - \frac{\partial\beta}{\partial h}\varphi = t_0$$
$$y_1 = 0, \quad y_2 = \frac{\alpha}{\sin\lambda} - \beta\operatorname{ctg}\lambda, \quad y_3 = \beta$$

are the integrals of the system. The functions $\xi(t)$ and $\eta(t)$ are obtained from the above expressions for $\xi^{\cdot}$ and $\eta^{\cdot}$ by means of quadratures.

The particular solution obtained corresponds to a motion in which the variables ψ and φ vary linearly with time, while ξ and η vary periodically with time. Since $\theta =$ const, the O_1z -axis describes a cone of revolution about the vertical passing through the disc center. Consequently the particular solution obtained corresponds to a uniform circular rolling of the disc. The latter case is discussed in detail in [6].

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PERIODIC SOLUTIONS FOR EQUATIONS OF CERTAIN AUTONOMOUS SYSTEMS

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The sufficient conditions are established for the existence of a stable limit cycle for systems of the form

$$x^{\cdot} = y, \quad y^{\cdot} = -g(x) - f_0(x)\, y - \ldots - f_n(x)\, y^n$$

for $n = 2$ and $n = 2m + 1$. The conditions of the theorem of existence and uniqueness of the solution are assumed to hold.

1. Consider the system

$$x^{\cdot} = y, \quad y^{\cdot} = -y\,[f_1(x)\, y + f_0(x)] - g(x) \tag{1.1}$$

introducing the notation

$$F_1(x) = \exp\left(\int_0^x f_1(x)\, dx\right), \quad F(x) = 2\int_0^x F_1(x) f_0(x)\, dx - \lambda \int_0^x \frac{dx}{F_1(x)}$$

$$r(x) = 2\int_0^x F_1^2(x)\, g(x)\, dx + \int_0^x F_1(x)\, F(x)\, f_0(x)\, dx$$

$$Q(x) = r(x) - \frac{1}{4} F^2(x), \quad G(x) = \int_0^x g(x)\, dx$$

Theorem 1. System (1.1) has at least one stable limit cycle, provided that the following conditions hold:

1. Numbers $a < b < 0 < c < d$ and $\lambda > 0$ exist such, that the functions $F(x)$ and $g(x)$ have the following consecutive signatures:

$$\begin{array}{llll} g(x) < 0 & \text{for } x \in (a, 0), & g(x) > 0 & \text{for } x \in (0, d) \\ F(x) < 0 & \text{for } x \in (a, b), & F(x) > 0 & \text{for } x \in (b, 0) \\ F(x) < 0 & \text{for } x \in (0, c), & F(x) > 0 & \text{for } x \in (c, d) \end{array}$$